Distribution of matchings in Myerson’s network formation game

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Abstract
Consider a population of \( n \) players playing a variant of Myerson’s network formation game. Each player simultaneously chooses \( k \) other players he would want to be connected to. If two players are in each other’s choice set, a matching occurs. We call the outcome of the network formation game a \( k \)-uniform Myerson graph and study the distribution of matchings on such graphs with homogenous and heterogeneous populations.

Keywords:
Matching distribution, Myerson graphs, Myerson’s network formation game, evolutionary game theory

1. Introduction

In most evolutionary models, the payoff of a player against the population is computed as the average payoff he obtains by playing with all other players (Kandori et al. (1993), Taylor and Jonker (1978), Taylor et al. (2004), Fudenberg et al. (2006)). This way of computation doesn’t take into consideration that a game is only played when two players consent to playing it. We propose therefore a model where mutual consent is required for game play.

In this model, each player offers a game to a subset of players. If two players offer each other a game, there is a matching, the game is played and payoffs are realized. Due to eventual mis-coordination, expected payoffs are typically lower than standard average payoffs. Generally, the payoff of each player depends on the number of the matchings in which he takes part. It suffices therefore to know the distribution of matchings in the population to derive the distribution of payoffs.

The setting where each player offers a game to a subset of the population and a game is played when both players consent to it is called Myerson’s network formation game (Jackson (2003), Van den Nouweland (2005)) and was first introduced by Myerson (1991). We call the outcome of Myerson’s network formation game a Myerson graph.

We compute the distribution of matchings on Myerson graphs when \( n \) players choose a subset of players of size \( k \leq n - 1 \) and show that the distribution of matchings is asymptotically Poisson. We study the distribution of matchings in a population with two types of players and provide an application for the results. This setting can be used to describe situations where players prefer to interact with a given type of players as in the case of assortative matching (Bergstrom (2003)). This paper is organized as follows. Section 2 presents the model. Section 3 discusses the results and section 4 concludes.

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2. The model
Let \( P \) a set of \( n \) players. The strategy of each player is a subset of the list of other players. For each player \( i \), \( s_i \subseteq P \setminus \{i\} \). Players simultaneously announce their strategies i.e. the players they want to connect to. There is matching between players \( i \) and \( j \) if and only if \( i \in s_j \) and \( j \in s_i \). We call the outcome of Myerson network formation game a Myerson Graph. Call \( M \) the set of all possible Myerson graphs.
Assume now that players choose exactly \( k \) other players. The strategy space of each player are the subsets of the set of other players of cardinality \( k \). We call the outcome of this network formation game a \( k \)-uniform Myerson graph.
Let \( M_{(k)} \) be the sample space and \( X \) the random variable counting the number of matchings on a \( k \)-uniform Myerson graph. We set, in what follows, to determine the distribution of \( X \) on \( M_{(k)} \).

3. Results
Note that if \( k = n − 1 \), each player offers a game to every player in the remaining population. There is only one Myerson graph in \( M_{(n−1)} \) and it is \( K_n \): the complete graph of order \( n \). \( X \) has a degenerate distribution and, for each player, the expected average payoff is equal to the standard average payoff. We treat, in what follows, the matching distribution on \( M_{(k)} \) with \( 1 \leq k < n − 1 \).

3.1. Matching distribution on \( M_{(1)} \)
The strategy of each player \( i \) is exactly one of the other players where all other players are equally likely to be selected so with probability \( 1/(n−1) \) each. A matching occurs between players \( i \) and \( j \) when \( s_i = \{j\} \) and \( s_j = \{i\} \). Note that the probability that exactly \( k \) matchings occur is the probability of having at least \( k \) matchings while the remaining \( n−2k \) players are left unmatched.
To determine these probabilities, we resort to classical results of graph theory. Let \( G \) be a simple graph of \( n \) nodes.
An \( r \)-matching in \( G \) is a set of \( r \) independent edges. The number of \( r \)-matchings in \( G \) will be denoted by \( \delta_r \). We set \( \delta_0 = 1 \) and define the matching polynomial of \( G \) by \( \zeta \) as \( \Sigma_{r=0}^{[n/2]} (-1)^r \delta_r x^{n−2r} \) where \([z]\) is the integer part of \( z \). On a complete graph, \( \delta_r = (-1)^r \frac{n^2 - r (n−2r)}{r!(n−2r)!} \) and the matching polynomial is equal to the Hermite polynomial (Farrell (1979)):

\[
\mathcal{H}_n(x) = \sum_{r=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^r \frac{n^2 - r}{r!(n−2r)!} x^{n−2r} \tag{1}
\]

**Lemma 1.** On \( M_{(1)} \), the probability of having no matchings is: \( (n−1)^{−n} \mathcal{H}_n(n−1) \).

**Proof.** Each node connects - with probability \( \frac{1}{n−1} \) - through a directed edge to one of the remaining \( n−1 \) nodes. A matching is then a directed cycle of order \( 2 \) and occurs with probability \( \frac{1}{(n−1)^2} \). Note that the probability of having at least \( k \) matchings is equal to \( \delta_k (n−1)^{n−2k} \); there are \( \delta_k \) ways to have \( k \) matchings while the remaining \( n−2k \) nodes can have any configuration. Using the inclusion-exclusion principle, the number of cases of exactly zero matchings is equal to the number of cases of at least zero matchings — the number of cases of at least one matching + the number of cases of at least two matchings etc. \# \( \{X = 0\} = \delta_0 (n−1)^{n−1} + \delta_1 (n−1)^{n−2} + \cdots + \delta_k (n−1)^{n−2k} \) = \( \mathcal{H}_n(n−1) \). There are \( (n−1)^n \) possible cases, therefore \( P[X = 0] = (n−1)^{−n} \mathcal{H}_n(n−1) \). \( \square \)
Following the reasoning of Lemma 1, the number of cases of exactly one matching equals \( \delta_n(1)_{\mathcal{H}_{n-2}}(n-1) \) i.e. for each of the \( \delta_n(1) \) cases of one matching occurring, no other matchings occur within the remaining \( n-2 \) individuals.

In general,

\[
P[X = k] = \frac{\delta_n(k) \times \mathcal{H}_{n-2k}(n-1)}{(n-1)^n} = \frac{(1/2)^k}{k!} \sum_{r=0}^{\lfloor k/2 \rfloor} \frac{(-1/2)^r n!(n-1)^{-2k-2r}}{r!(n-2k-2r)!}
\]  

(2)

Let \( \mathcal{X}_{(1)} \) be the distribution of matchings on \( \mathcal{M}_{(1)} \). How does \( \mathcal{X}_{(1)} \) behave when \( n \to \infty \)?

Note that \( E[X] = \frac{(n-1)}{2} \times \frac{1}{(n-1)} = \frac{n}{2(n-1)} \) converges from above to \( \frac{1}{2} \) when \( n \to \infty \). Indeed, we find that \( \mathcal{X}_{(1)} \) converge in distribution to a Poisson with mean \( \lambda = 1/2 \).

**Proposition 1.** The distribution of matchings on \( \mathcal{M}_{(1)} \) is approximately Poisson with mean \( \lambda = 1/2 \) when \( n \to \infty \).

**Proof.** See Appendix.

The result is a special case of the results of Erdos and Rényi (1960) and has some similarity with the generalization of the matching distribution of Niermann (1999). However, the convergence is slower in our case.

**Application.** A logical matrix is a matrix whose entries are either 0 or 1. Consider the set of logical matrices \( \mathcal{M} \) that have exactly one non-zero element (1) in each row and whose diagonal entries are zeros. A matching occurs when \( i \to j \) and \( j \to i \). If we place the choice of player \( i \) in row \( i \), we have a matching when \( M_{ij} = M_{ji} \). The matrix of size \( n \times n \) is symmetric if we have \( \frac{n}{2} \) matchings. The number of logical matrices of size \( n \times n \) with diagonals of zero and containing a symmetric matrix of size \( k \times k \) is equal to the number of configurations that \( k \) matchings occur. Therefore the distribution of matrices of size \( n \times n \) containing a symmetric \( k \times k \) matrix is \( \mathcal{X}_{(1)} \) for finite \( n \) and approximately Poisson (1/2) when \( n \) goes to \( \infty \).

3.2. Matching distributions on \( \mathcal{M}_{(1)} \) with 2 types

Above, we considered the case where players are identical and the probability of selecting each other player is the same. We relax the assumption here. Players can be of two types \( \{1, 2\} \). There are \( n_1 \) type 1 players and \( n_2 \) type 2 players. There are three kinds of matchings: \( \tau_1 \) between two type 1 players, \( \tau_2 \) between two type 2 players and \( \tau_3 \) between one player of type 1 and one player of type 2, occurring respectively with probability \( p_1, p_2 \) and \( p_3 \). Let \( X_i \) be the random variable counting the number of matchings of type \( \tau_i \). Let \( X = \{X_1, X_2, X_3\} \), \( p = \{p_1, p_2, p_3\} \) and \( k = \{k_1, k_2, k_3\} \).

Our strategy to compute the probability of having exactly \( k_i \) matchings of type \( \tau_i \) is to compute first the probability of having at least \( k_i \) matchings of type \( \tau_i \), which we multiply with the probability of having zero matchings among the remaining players.

**Lemma 2.** The number of distinct 1-uniform Myerson graphs with \( k_i \) matchings of type \( \tau_i \) for \( i = 1, 2, 3 \) is:

\[
\mu_{n_1,n_2}(k) = \frac{2^{-k_1-k_2}}{k_1!k_2!k_3!} \frac{n_1!}{(n_1-2k_1-k_3)!} \frac{n_2!}{(n_2-2k_2-k_3)!}
\]

**Proof.** See Appendix.
Define now the signed matching generating polynomial $\Gamma_{n_1, n_2}(x)$ by:

$$\Gamma_{n_1, n_2}(x) = \sum_{r_1=0}^{\lfloor \frac{n_1}{2} \rfloor} \sum_{r_2=0}^{\lfloor \frac{n_2}{2} \rfloor} \sum_{r_3=0}^{\min\{n_1-2r_1, n_2-2r_2\}} (-1)^{\sum r} \mu_{n_1, n_2}(r) x_1^{r_1} x_2^{r_2} x_3^{r_3}$$  \(3\)

**Lemma 3.** If the matching of type $\tau_i$ occurs with probability $p_i$ for $i = 1, 2, 3$ then the probability of having no matchings on $M_{(1)}$ is equal to $\Gamma_{n_1, n_2}(p)$.

*Proof.* See Appendix.

Combining the results of Lemma 2 and Lemma 3, we can determine the distribution of matchings on $M_{(1)}$.

**Proposition 2.** The probability of having exactly $k_i$ matchings of type $\tau_i$ for $i = 1, 2, 3$ is equal to:

$$P[X = \mathbf{k}] = \mu_{n_1, n_2}(k) p_1^{k_1} p_2^{k_2} p_3^{k_3} \Gamma_{n_1-2k_1-k_3, n_2-2k_2-k_3}(p)$$

*Proof.* The probability of having at least $k_i$ matchings of type $\tau_i$ is $\mu_{n_1, n_2}(k) p_1^{k_1} p_2^{k_2} p_3^{k_3}$. If the remaining players are unmatched, then we have exactly $k_i$ matchings of type $\tau_i$. This occurs with probability $\Gamma_{n_1-2k_1-k_3, n_2-2k_2-k_3}(p)$.

Using this same reasoning, we can generalize the results for an arbitrary number of types.

**Application.** We illustrate this result by a game between cooperators and defectors that are imperfectly labelled (Bergstrom (2003), Ghachem (2016)). Consider a population of size $n$ with $i$ cooperators and $n-i$ defectors playing a prisoners’ dilemma with the following payoff matrix:

$$\begin{bmatrix}
    b & -c \\
    -c & 0
\end{bmatrix}$$

In a prisoners’ dilemma game, both cooperators and defectors prefer to be matched with cooperators. If players’ strategies are observable with perfect accuracy, then cooperators will only play with cooperators. Detection is typically less than perfectly accurate. We assume that a cooperator is labelled as an apparent cooperator with a probability $(1-\alpha)$ and a defector as an apparent cooperator with a probability of $\beta$ with $1-\alpha \geq \beta$. Given that $1-\alpha \geq \beta$, both cooperators and defectors would like to condition their partner choice on the label.

Individuals have an exogenous search capacity $r$. Initially, each individual samples uniformly and randomly a partner from the population. If the sampled partner is labelled, the individual stops searching and offers a game to the labelled partner. If the sampled partner is unlabelled, the individual continues sampling (with replacement) until a labelled individual is drawn or the search capacity $r$ is exhausted. If the search capacity is exhausted, the most recently drawn partner is offered a game. A game occurs if and only if there is a match, i.e., two individuals pick each other in a
single round. Let \( p_{(r)} \) be the probability that a cooperator selects a labelled partner after \( r \) attempts with:

\[
p_{(r)} = 1 - \left( \frac{n - l(i)}{n - 1} \right)^r \tag{4}
\]

where \( l(i) = (1 - \alpha)(i - 1) + \beta(n - i) \). For a cooperator, a labelled partner is a cooperator with probability \( c(i) = [(1 - \alpha)(i - 1)]/l(i) \) and a defector \( 1 - c(i - 1) \); and an unlabelled partner is a cooperator with probability \( c_d(i - 1) = [\alpha(i - 1)]/[n - 1 - l(i)] \) and a defector with probability \( 1 - c_d(i - 1) \). The probability that a cooperator draws a cooperator after \( r \) attempts is equal to:

\[
p_{cc} = p_{(r)} c(i) + (1 - p_{(r)}) c_d(i) \tag{5}
\]

The probabilities \( p_{cd}, p_{dc} \) and \( p_{dd} \) are defined in a similar fashion. A game between two cooperators occurs with probability \( p_{cc}^2 \); between two defectors with probability \( p_{dd}^2 \) and between a cooperator and a defector with probability \( p_{cd}p_{dc} \); so \( p = \{p_{cc}^2, p_{dd}^2, p_{cd}p_{dc}\} \).

Assuming that \( i \geq 2 \) and \( n - i \geq 4 \); the probability to simultaneously have one game between two cooperators and 2 games among defectors and zero games between a cooperator and a defector is equal to:

\[
P[X = [1, 2, 0]] = \mu_{n-i}(1, 2, 0) p_{cc}^2 p_{dd}^4 (p_{cd}p_{dc})^0 \Gamma_{1-2} \Gamma_{n-i-4}(p) \tag{6}
\]

On this Myerson graph, the average payoff of a cooperator is \((b - c)/(n - 1)\) and the average payoff of a defector is 0. In general, the average payoff of a cooperator \((\pi_c)\) and of a defector \((\pi_d)\) on a given graph \( \gamma_k \) with \( k \) matchings of type \( \tau \) are respectively:

\[
\pi_c = \frac{(b - c)k_1 + (-c)k_3}{n - 1} \quad \pi_d = \frac{bk_3}{n - 1}
\]

### 3.3. Matching distribution on \( \mathcal{M}_{(n-2)} \)

On \( \mathcal{M}_{(n-1)} \), all possible matchings are realized and we have \( \frac{n(n-1)}{2} \) matchings. Note that by randomly deleting one directed edge from each node of an \((n - 1)\)-uniform Myerson graph, we obtain an \((n - 2)\)-uniform Myerson graph.

In other words, each \( g \in \mathcal{M}_{(n-2)} \) has its complement \( \overline{g} \in \mathcal{M}_1 \). If \( \overline{g} \) has \( k \) matchings, \( K_n \) loses \( n - k \) matchings and \( g \) has therefore \( \frac{n(n-1)}{2} - (n-k) \) matchings. Of course, the minimum (maximum) number of matchings in \( \overline{g} \) is \( 0(n/2) \).

Therefore, the maximum number of matchings on \( \mathcal{M}_{(n-2)} \) is \( \frac{n(n-2)}{2} \) and the minimum is \( \frac{n(n-3)}{2} \).

Using (2), the distribution of matchings on \( \mathcal{M}_{(n-2)} \) is given by:

\[
P \left[ X = \frac{n(n-3)}{2} + k \right] = \frac{(1/2k)^{\frac{k-2}{2}}}{k!} \sum_{r=0}^{\frac{k-2}{2}} \frac{(-1/2)^r}{r!} \frac{n!(n-1)^{2k-2r}}{(n-2k-2r)!} \tag{7}
\]

if \( 0 \leq k \leq \frac{n}{2} \) and zero otherwise.
3.4. Matching distribution on $M_{(k)}$

To derive analytical results for the distribution of $M_{(k)}$, where $k > 1$ is very complex. We can still show, using the Chen-Stein method (Arratia et al. (1990)), that the distribution of matchings is approximately Poisson with mean $k^2/2$ when $k \ll n$.

4. Conclusion

We study the distribution of matchings on $k$-uniform Myerson graphs for $1 \leq k < n$ with homogenous and heterogenous populations with the purpose of giving a credible basis for the computation of payoffs in an evolutionary setting. By introducing mutual consent, the model adds a layer of realism to payoff calculation. It exhibits, however, high level of mis-coordination rarely observed in reality, especially when $k \ll n$. To curb uncertainty and ensure coordination, people not only use coordination mechanisms (reinforcement, communication and promises), but usually restrict the set of their potential partners. An interesting extension of the paper would be to study Myerson’s network formation game on an $m$-regular graph i.e. where each player has $m$ neighbors and offers a game to $k$ of his $m$ neighbors with $k \leq m$. If $k = m$, the unique Myerson graph is the $m$-regular graph and average payoffs are computed as in the standard way.
Appendix

**Proposition 1.** The distribution of matchings on $\mathcal{M}_{(1)}$ is approximately Poisson of mean $\lambda = 1/2$ when $n \to \infty$.

*Proof.* We rely on Chen Stein Poisson approximation and apply Theorem 1 in Arratia et al. (1989). Here the index set $I$ is the set of edges and $X_n$ is the indicator that there is a matching between the two end nodes of edge $\alpha$ with $p_\alpha = \frac{1}{(n-1)^2}$ and $\lambda = \frac{2n}{2(n-1)}$. We take $B_\alpha$ to be the set of all edges sharing a node with $\alpha$, so $b_2 = b_3 = 0$ and $b_1 = |I\setminus B_\alpha| p_\alpha^n = \frac{n(n-1)}{2(n-1)^2}(2(n-2))\frac{1}{2(n-1)^2} = \frac{n(n-2)}{(n-1)^2}$. Theorem 1 states that the distance between a Poisson of mean $\frac{n}{2(n-1)}$ and $X$ is smaller than $2b_1 = \frac{2(n-2)}{(n-1)^2}$. When $n \to \infty$, $b_1 \to 0$ and $\lambda \to \frac{1}{2}$. $X$ is approximately Poisson with mean $\frac{1}{2}$ when $n \to \infty$. \hfill \Box

**Lemma 2.** The number of distinct $1$-uniform Myerson graphs with $k_i$ matchings of type $\tau_i$, for $i = 1, 2, 3$ is:

$$\mu_{n_1,n_2}(k) = \frac{2^{-k_1-k_2}}{k_1!k_2!k_3!} \frac{n_1!}{(n_1-2k_1-k_3)!} \frac{n_2!}{(n_2-2k_2-k_3)!}.$$  

*Proof.* Since we have $k_1$ matchings of type $\tau_1$ and $k_2$ matchings of type $\tau_2$, there are $n_1 - 2k_1$ type 1 unmatched players and $n_2 - 2k_2$ type 2 unmatched players. These players can’t be matched with the same type players. They form then a bipartite graph. The number of possible matchings among them is given by classical results in graph theory. The number of $r$ matchings on a bipartite graph $K_{n_1,n_2}$ is equal to $\phi_{n_1,n_2}(r) = \frac{n_1!}{r! (n_1-r)!} \frac{n_2!}{(n_2-r)!}$ with $r \in [0, \min(n_1, n_2)]$. We have then $\mu_{n_1,n_2}(k) = \delta_{n_1}(k_1) \delta_{n_2}(k_2) \phi_{n_1-2k_1,n_2-2k_2}(k_3)$, and therefore:

$$\mu_{n_1,n_2}(k) = \frac{2^{-k_1-k_2}}{k_1!k_2!k_3!} \frac{n_1!}{(n_1-2k_1-k_3)!} \frac{n_2!}{(n_2-2k_2-k_3)!}.$$  

Of course, $\mu_{n_1,n_2}(0) = \delta_{n_1}(0) \delta_{n_2}(0) \phi_{n_1-2k_1,n_2-2k_2}(0) = 1$. \hfill \Box

**Lemma 3.** If the matching of type $\tau_i$ occurs with probability $p_i$ for $i = 1, 2, 3$ then the probability of having no matchings on $\mathcal{M}_{(1)}$ is equal to $\Gamma_{n_1,n_2}(p)$.

*Proof.* The event of having least $(k_1, k_2, k_3)$ matchings has the probability $p_{1}^{k_1}p_{2}^{k_2}p_{3}^{k_3}$. We use the probabilistic version of the inclusion-exclusion principle. The probability of having exactly zero matchings is equal to the probability of having at least zero matchings — the probability of having one matching (3 cases) + the probability of having two matchings (6 cases) $\ldots$$P[X = 0] = \mu_{n_1,n_2}(0,0,0)p_{1}^{0}p_{2}^{0}p_{3}^{0} - \mu_{n_1,n_2}(1,0,0)p_{1}^{1}p_{2}^{0}p_{3}^{0} - \mu_{n_1,n_2}(0,1,0)p_{1}^{0}p_{2}^{1}p_{3}^{0} \ldots$.

$$P[X = 0] = \sum_{i_1=0}^{\lfloor \sqrt{2n} \rfloor} \sum_{i_2=0}^{\min[n_1-2i_1,n_2-2i_2]} \sum_{i_3=0}^{\lfloor n_3/2 \rfloor} (-1)^{i_1} \times \mu_{n_1,n_2}(i_1,i_2,i_3)p_{1}^{i_1}p_{2}^{i_2}p_{3}^{i_3} = \Gamma_{n_1,n_2}(p)$$  

\hfill \Box

7
References


