Prudent coopersators can save cooperation

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Abstract
In the prisoners dilemma, cooperators are at an obvious disadvantage compared to defectors in well-mixed populations. We extend the game by introducing a new strategy prudent cooperation where cooperators pay a cost to identify the strategy of their opponents and abstain from playing the game if matched with a defector. We study the strategic nature of the resulting game and identify different cost thresholds that determine the behavior of the dynamics. If the cost of detection of defectors is sufficiently low, the game is a rock-paper-scissors game and cooperation can evolve with the help of prudent cooperators.

Keywords: Prisoners’ dilemma, reputation, rock-paper-scissors, ESS, replicator dynamics, indirect reciprocity

1. Introduction

Indirect reciprocity is one of the major mechanisms of the evolution of cooperation (Nowak (2006)). This mechanism relies on the idea that people with good reputation are more likely to be helped than those with bad reputation (Nowak and Sigmund (1998b), Nowak and Sigmund (1998a)). An extensive body of research, both experimental and theoretical, has been conducted to explore the inner workings of this mechanism (Engelmann and Fischbacher (2009), Dufwenberg et al (2001), Fishman (2003), Bshary and Grutter (2006), Brandt and Sigmund (2006), Brandt and Sigmund (2005), Brandt and Sigmund (2004), Berger and Grne (2014), Berger (2011), Aktipis (2004), Leimar and Hammerstein (2001), Mailath and Samuelson (2007), Matsuo et al (2014), Milinski et al (2002), Nakamura and Masuda (2011), Nowak and Sigmund (2005), Ohtsuki and Iwasa (2006), Ohtsuki et al (2009), Panchanathan and Boyd (2003), Roberts (2008), Olejarz et al (2015), Ghang and Nowak (2015)). In the classical models, a strategy for playing indirect reciprocity consists of an action rule and a social norm. The social norms assesses the reputation of the interaction partner and the action rule determines the action given this information (Nowak and Sigmund (2005)). In the action rule, players choose either to cooperate or defect i.e. it assumes obligatory interactions. A new avenue of research has explored the possibility of optional interactions in the framework of indirect reciprocity (Ghang and Nowak (2015), Ghachem (2016)). In this setting, a player can abstain from interacting with a defector, once he discovers his identity. There are many real situations where individuals refuse to interact with presumed defectors. Banks refuse to give loans to possible defaulters while loan applicants refuse loan offers from banks known for their hidden

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costs. Ghang and Nowak (2015) relate the evolution of cooperation to the probability of discovering the reputation of a defector. Here, we take another path and link the evolution of cooperation to the cost of acquiring information about the strategy of the opponent. We assume there exists an institution that produces information about the reputation of individuals and makes it available to individuals for a cost. Through this assumption, we avoid the challenges related to the social norm assessment. An observer seeing a cooperator defecting on a defector can label the former as a defector i.e. the context matters (Sugden (1986), Kandori (1992), Panchanathan and Boyd (2003), Ohtsuki and Iwasa (2006), Ohtsuki et al (2009), Matsuo et al (2014)).

For simplicity, we assume that the information produced by the institution is truthful and study the game where cooperators, who acquired the information, abstain from interacting with defectors. We do this by introducing a new strategy called prudent cooperation. This strategy will be available to players in addition to the standard strategies of cooperation and defection. A prudent cooperator pays a cost \( c \) to discover the strategy of his opponent. If his opponent is a cooperator (or a prudent cooperator), he plays the game; otherwise, he abstains. We study the strategic nature of the resulting game. If the cost is prohibitively high, it doesn’t pay off to investigate the strategy of the opponent and defectors take over the population (Nowak (2006)). We identify a cut-off cost such that if detection cost is below this cut-off level, the resulting game is a rock-paper-scissors game and they dynamics converges to a state where cooperation, defection and prudent cooperation coexist. The lower this cut-off level, the lower is the equilibrium fraction of defectors. This equilibrium is, however, not ESS and can be escaped through neutral drift. However, given that the perturbations and the detection cost are sufficiently small, the system recovers its equilibrium position, given that the equilibrium is asymptotically stable. Typically, the information about the strategy of individuals in society are produced by institutions and disseminated in society e.g. rating agencies. If the information is accessible to cooperators at a sufficiently low cost, prudent cooperators would be able to stop defectors and cooperation evolves.

The paper is organized as follows. Section 2 introduces the model, section 3 discusses the strategic nature of the game, section 4 focuses on the case of the rock-paper-scissors game and section 5 concludes.

2. The model

In the prisoners’ dilemma, two players have the choice to cooperate or defect. Both obtain payoﬀ \( R \) if there is mutual cooperation, but a lower payoﬀ \( P \) if there is mutual defection. If one individual defects, while the other cooperates, then the defector receives the highest payoﬀ \( T \), whereas the cooperator receives the lowest payoﬀ \( S \). We have \( T > R > P > S \). It is well established that defection dominates cooperation in mixed populations (Nowak (2006)). We introduce a new strategy prudent cooperation (PC). The user of such a strategy is called a prudent cooperator. A prudent cooperator acquires information about his interaction partner and refrains from playing if he suspects his partner to be a defector. For simplicity of argument, we assume that this prudent cooperator can spend an amount \( c \) to correctly identify the strategy of his partner. If his partner is a cooperator, he plays the game and gets \( R \); otherwise, he does not play and gets 0. We call the resulting game \( G \). In the presence of these three strategies, the payoﬀ matrix of the game becomes:

We study the evolution of the strategies using replicator dynamics. Strategies are inherited and spread proportionally to their success, where success is measured in terms of fitness. The population states will be identified with the elements of the two-dimensional simplex \( \Delta \) with \( \Delta := \{p \in \mathbb{R}^3 | \sum_{i=1}^{3} p_i = 1 \} \) where \( p_1, p_2 \) and \( p_3 \) are respectively
the frequency of cooperators, defectors and prudent cooperators in the population. Assuming random matching, the individual payoff of an $i$-strategist at a population state $p$ is given by:

$$\pi_i(p) = \sum a_{ij}p_j = (Ap)_i$$  \hspace{1cm} (1)

The average payoff of the population at state $p$ is:

$$\bar{\pi}(p) = \sum p_i \pi_i(p)$$  \hspace{1cm} (2)

The relative fitness or excess payoff of strategy $i$ at state $p$ is given by $

$$\pi_i(p) - \bar{\pi}(p).$$

The continuous replicator dynamics is a system of differential equations on the strategy simplex $\Delta$:

$$\dot{p}_i = p_i(\pi_i(p) - \bar{\pi}(p))$$  \hspace{1cm} (3)

A rest point $p^*$ of the replicator dynamics is a stationary solution of (3) satisfying $\dot{p}_i = 0 \forall i$. Any pure strategy or state is clearly a rest point. For a completely mixed rest point $p^*$, the payoffs of each strategy in the support of $p^*$ are the same. Since the orbits and trajectories in the replicator dynamics are invariant in response to local shifts in payoffs (Weibull (1997)), we transform the game $G$ into its essential form $G_c$ (Weissing (1991)). Its payoff matrix $A_c$ is given by Figure 2(a). By defining $g_{i\rightarrow j}$ in matrix $A_c$ as the gain from switching from strategy $i$ to strategy $j$ when meeting strategy $i$ with $g_{i\rightarrow i} = 0$, we can rewrite the matrix $A_c$ as in Figure 2(b).

We will restrict our attention to the case where $R > 0$ and $S < 0$. Prudent cooperators should have incentive to play with cooperators ($R > 0$) and gain by not playing with defectors ($S < 0$). Given $T > R > P > S$ and $c > 0$, all signs except those of $c - R$ and $-c - P$ are defined. Let $s_1$ be the sign of $-c - P$ and $s_2$ be the sign of $c - R$. The sign pattern of $A_c$ is given by Figure 3 (Hofbauer and Sigmund (1998)). In the next section, we characterize equilibria of

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\[ \begin{bmatrix} C & D & PC \\ C & R & S & R \\ D & T & P & 0 \\ PC & R - c & -c & R - c \end{bmatrix} \]

Figure 1: Payoff matrix $A$ of the game $G$

\[ \begin{bmatrix} C & D & PC \\ C & 0 & S - P & c \\ D & T & R & 0 \\ PC & -c - R & -c - P & 0 \end{bmatrix} \quad \begin{bmatrix} C & D & PC \\ C & 0 & g_{D\rightarrow C} & g_{PC\rightarrow C} \\ D & g_{C\rightarrow D} & 0 & g_{PC\rightarrow D} \\ PC & g_{C\rightarrow PC} & g_{D\rightarrow PC} & 0 \end{bmatrix} \]

(a) The interactive payoff matrix $A_c$ (b) $A_c$ in terms of gains of switching

Figure 2: Two different representations of the interactive payoff matrix $A_c$

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\[ A_0 = \begin{bmatrix} R & P & R \cdot c \\ R & P & R \cdot c \end{bmatrix} \]

We decompose $A$ into matrices: $A_c$ is called the interactive component of $A$ and $A_1$ is called the non-interactive component of $A$. The matrix $A_0$ has zeroes along its diagonal and is obtained by subtracting $A_1$ from $A$ where $A_1 = \begin{bmatrix} R & P & R \cdot c \\ R & P & R \cdot c \end{bmatrix}$. }

3
the game \( G_c \) for the different scenarios of \((s_1, s_2)\). We emphasize the case where \((s_1, s_2) = (-, +) \) \((G_c \) is a generalized rock-paper-scissors game\) which we study thoroughly in Section 4.

\[
\begin{bmatrix}
C & D & PC \\
0 & 0 & - \\
+ & 0 & s_2 \\
PC & s_1 & 0 \\
\end{bmatrix}
\]

Figure 3: The sign pattern of payoff matrix \( A_c \)

3. Strategic nature of the game and equilibrium characterization

Since the game restricted to the strategies C and D is a prisoners’ dilemma so there can’t be an equilibrium putting positive weights on C and D and zero on PC. The same holds for the game restricted on C and PC. Also, \((1, 0, 0)\) is not an equilibrium profile since D is a strict best response to C. Similarly, for the strategy profile \((0, 0, 1)\) since C is a strict best reply to PC. For generic payoffs, there are only three possible equilibria:

\[
p^*_1 : (0, 1, 0) \quad p^*_2 : \left(0, \frac{R - c + P + c}{P + R}, \frac{P + c}{P + R}\right) \quad p^*_3 : \left(\frac{RS - c(S - P - R)}{ST}, \frac{-c}{S}, \frac{c(S + T - P - R) - RS + ST}{ST}\right)
\]

At the interior rest point \( p^*_3 \), the equilibrium fraction of defectors is negatively correlated with the cost of identifying defectors. If prudent cooperators easily identify defectors \( (c \) is small) then their equilibrium fraction is low. Interestingly, this fraction is also decreasing in the absolute value of the sucker payoff \(|S|\). It is therefore in the interest of defectors not to greatly abuse cooperators so that they can be more numerous in the equilibrium. This is reminiscent of viruses controlling their virulence so as not to kill their hosts and survive (Ewald (1996)).

Note that C gains a higher payoff against \( p^*_2 \) than \( p^*_2 \) gains against itself when \( c > \frac{RS}{S - P - R} \). Call \( \bar{c} = \frac{RS}{S - P - R} \) and \( NE(G_c) \) the set of Nash equilibria of the game \( G_c \). We have then:

\[
c > \bar{c} \Rightarrow p^*_2 \notin NE(G_c) \quad c < \bar{c} \Rightarrow p^*_2 \in NE(G_c)
\]

We can easily verify that whenever \( c < \bar{c}, p^*_2 \in \Delta \) and \( p^*_3 \in \Delta \). For generic payoffs, the signs of \( s_1 \) and \( s_2 \) will determine the strategic nature of the game. To study the stability of equilibria, we will make use of the following theorem (Weissing, 1991, Theorem 5.1).

**Theorem 1.** If \( p^* \) is an ESS of \( A_c \), it is asymptotically stable and a global attractor of (3).

3.1. **Scenario I:** \((s_1, s_2) = (+, -)\)

This occurs if \( c > \max[-P, R] \). The cost of identifying defectors is so prohibitively high that prudent cooperators fare worse than defectors. Neither cooperators, nor prudent cooperators can stop defectors from taking over the population. The sign pattern of \( A_c \) is given in Figure 4.

In this case, the game is dominance solvable: D dominates PC. After eliminating PC, D dominates C. It follows that \( p^*_1 \) is a strict Nash equilibrium and therefore an ESS. Since any ESS is an asymptotically rest point in the replicator dynamics, we deduce that \( p^*_1 \) is an asymptotically stable rest point. The two other vertices of the simplex are also rest
Figure 4: The sign pattern of payoff matrix $A_c | (s_1, s_2) = (+, -)$

points of the replicator dynamics. For payoff entries $T = 3; R = 2; P = -1; S = -2$ and a cost $c = 3$, Figure 5 illustrates the phase diagram for the replicator dynamics.

Figure 5: Phase diagram for the replicator dynamics $|(s_1, s_2) = (+, -)| T = 3; R = 2; P = -1; S = -2$ and $c = 3$

3.2. Scenario II: $(s_1, s_2) = (-, -)$

This occurs whenever $c \in (-P, R)$. Since $S < 0$, we have $-P < \tilde{c} < R$. If the cost is relatively high ($\tilde{c} < c < R$), then $PC$ is strictly dominated by a convex combination of $C$ and $D$. Once $PC$ is eliminated, $D$ dominates $C$. If $c < \tilde{c}$, the game has three equilibria but only $D$ is asymptotically stable. We do not require $P$ to be negative, just that $-P < R$.

The sign pattern of $A_c$ is given by Figure 6.

Figure 6: The sign pattern of payoff matrix $A_c | (s_1, s_2) = (-, -)$

We identify two cases:

- **Case I**: $c \in (\tilde{c}, R)$
  
  If $c > \tilde{c}$ then the interval $(\frac{c-P}{R}, \frac{-c-P}{R})$ is not empty. For any $\omega \in (\frac{c-P}{R}, \frac{-c-P}{R})$, the mixed strategy $\omega C + (1 - \omega) D$ dominates $PC$. After eliminating $PC$, $D$ dominates $C$ and again $p_2^*$ is a strict equilibrium and therefore an ESS. In the replicator dynamics, all vertices are rest points while $(0, 1, 0)$ is the unique asymptotically stable, since it is an ESS (Theorem 1,(Hofbauer and Sigmund, 1998, Theorem 7.2.4)). Note that in the absence of cooperators, $p_2^*$ is NE and therefore a rest point of the replicator dynamics. It is however unstable. For payoff
entries $T = 3; R = 2; P = -1; S = -2$ and a cost $c = \frac{5}{4} \in (\bar{c} = \frac{4}{3}, 2);$ Figure 7 illustrates the phase diagram for the replicator dynamics.

\begin{center}
\includegraphics[width=0.5\textwidth]{phase_diagram.png}
\end{center}

**Figure 7: Phase diagram for the replicator dynamics**

\begin{align*}
&\text{Case 2: } c \in (-P, \bar{c}) \\
&\text{We check whether the equilibria } (p_1^*, p_2^*, p_3^*) \text{ are Nash equilibria of } G_c. \text{ From Figure 6, we see that } D \text{ is a strict best response to itself and conclude that } p_1^* \text{ is a strict Nash equilibrium and, therefore, an ESS. From (5), the equilibrium } p_2^* \text{ is also a Nash equilibrium since } c < \bar{c}. \text{ Note that the equilibrium payoff at } p_2^* \text{ is negative. Since } PC \text{ is in the support of } p_2^*, \text{ it earns the same as } p_2^* \text{ against } p_2^*. \text{ However, } p_2^* \text{ gets a negative payoff against } PC \text{ while } PC \text{ gets zero against itself. We conclude that } p_2^* \text{ is not an ESS. Since the number of equilibria in a generic game is odd, therefore } p_3^* \text{ is also a Nash equilibrium of } G_c. \text{ Of course, all NE are rest points for the replicator dynamics (Hofbauer and Sigmund (1998)). However, the unique asymptotically stable rest point is } p_3^*, \text{ since it is an ESS. For payoff entries } T = 3; R = 2; P = -1; S = -2 \text{ and a cost } c = \frac{5}{4} \in (1, \bar{c} = \frac{4}{3}), \text{ Figure 8 illustrates the phase diagram for the replicator dynamics.}
\end{align*}

\begin{center}
\includegraphics[width=0.5\textwidth]{phase_diagram2.png}
\end{center}

**Figure 8: Phase diagram for the replicator dynamics**

\begin{align*}
&\text{Case 2: } c \in (-P, \bar{c}) \\
&\text{We check whether the equilibria } (p_1^*, p_2^*, p_3^*) \text{ are Nash equilibria of } G_c. \text{ From Figure 6, we see that } D \text{ is a strict best response to itself and conclude that } p_1^* \text{ is a strict Nash equilibrium and, therefore, an ESS. From (5), the equilibrium } p_2^* \text{ is also a Nash equilibrium since } c < \bar{c}. \text{ Note that the equilibrium payoff at } p_2^* \text{ is negative. Since } PC \text{ is in the support of } p_2^*, \text{ it earns the same as } p_2^* \text{ against } p_2^*. \text{ However, } p_2^* \text{ gets a negative payoff against } PC \text{ while } PC \text{ gets zero against itself. We conclude that } p_2^* \text{ is not an ESS. Since the number of equilibria in a generic game is odd, therefore } p_3^* \text{ is also a Nash equilibrium of } G_c. \text{ Of course, all NE are rest points for the replicator dynamics (Hofbauer and Sigmund (1998)). However, the unique asymptotically stable rest point is } p_3^*, \text{ since it is an ESS. For payoff entries } T = 3; R = 2; P = -1; S = -2 \text{ and a cost } c = \frac{5}{4} \in (1, \bar{c} = \frac{4}{3}), \text{ Figure 8 illustrates the phase diagram for the replicator dynamics.}
\end{align*}

\begin{center}
\includegraphics[width=0.5\textwidth]{phase_diagram2.png}
\end{center}

**Figure 8: Phase diagram for the replicator dynamics**

\begin{align*}
&\text{Case 2: } c \in (-P, \bar{c}) \\
&\text{We check whether the equilibria } (p_1^*, p_2^*, p_3^*) \text{ are Nash equilibria of } G_c. \text{ From Figure 6, we see that } D \text{ is a strict best response to itself and conclude that } p_1^* \text{ is a strict Nash equilibrium and, therefore, an ESS. From (5), the equilibrium } p_2^* \text{ is also a Nash equilibrium since } c < \bar{c}. \text{ Note that the equilibrium payoff at } p_2^* \text{ is negative. Since } PC \text{ is in the support of } p_2^*, \text{ it earns the same as } p_2^* \text{ against } p_2^*. \text{ However, } p_2^* \text{ gets a negative payoff against } PC \text{ while } PC \text{ gets zero against itself. We conclude that } p_2^* \text{ is not an ESS. Since the number of equilibria in a generic game is odd, therefore } p_3^* \text{ is also a Nash equilibrium of } G_c. \text{ Of course, all NE are rest points for the replicator dynamics (Hofbauer and Sigmund (1998)). However, the unique asymptotically stable rest point is } p_3^*, \text{ since it is an ESS. For payoff entries } T = 3; R = 2; P = -1; S = -2 \text{ and a cost } c = \frac{5}{4} \in (1, \bar{c} = \frac{4}{3}), \text{ Figure 8 illustrates the phase diagram for the replicator dynamics.}
\end{align*}

\begin{center}
\includegraphics[width=0.5\textwidth]{phase_diagram2.png}
\end{center}

**Figure 8: Phase diagram for the replicator dynamics**

3.3. **Scenario III:** $(s_1, s_2) = (+, +)$

This occurs if $c \in (R, -P)$. Since $R > 0$, this case is only possible if $P < 0$. In this scenario, the strategy profile $p_1^*$ is not a Nash equilibrium. It is easy to check that if $P < -P$, then $R < \bar{c} < -P$. If $\bar{c} < c < -P$, then the unique Nash equilibrium is $p_2^*$ and defectors and prudent cooperators coexist. If the cost is relatively low ($R < c < \bar{c}$), then the unique Nash equilibrium is $p_3^*$ where all strategies coexist. The sign pattern of $A_c$ is given in Figure 9.
Here, $D$ is not a best response to itself for the parameter range, therefore $p_2^*$ is not a Nash equilibrium. Since the game is generic, either $p_2^*$ or $p_3^*$ are equilibria. We know, from (5), that $p_2^*$ is a Nash equilibrium only if $c > \bar{c}$. If $c < \bar{c}$, $p_2^*$ is not an equilibrium and therefore $p_3^*$ is. The two cases are:

- **Case 1: $c \in (\bar{c}, -P)$**

  From (5), strategy $C$ is not a best response to $p_2^*$ and therefore $p_2^*$ is the unique Nash equilibrium. The equilibrium $p_2^*$ is also ESS: (1) $C$ is not a best response to $p_2^*$ so earns less against $p_2^*$ than $p_3^*$ against itself. (2) The strategies $D$ and $PC$ are in the support of $p_2^*$ so earn against $p_2^*$ as $p_2^*$ earns against itself. However, $p_2^*$ earns a higher payoff against both $D$ and $PC$ then they earn against themselves. Since $p_2^*$ is an ESS, so is also asymptotically stable (Theorem 1). Despite the fact that $p_2^*$ is an evolutionary and asymptotically stable state where cooperation evolves; it is not interesting because the equilibrium payoff at $p_2^*$ is negative. Prudent cooperators might achieve a higher payoff if they abstain from playing altogether and gain 0 thereby. For payoff entries $T = 2; R = 1; P = -3; S = -4$ and a cost $c = \frac{5}{2} \in (\bar{c} = 2, 3)$.

- **Case 2: $c \in (R, \bar{c})$**

  $G_c$ is a generalized rock-paper-scissors game\(^2\). Since $c < \bar{c}$, strategy $C$ is a best response to $p_2^*$ and therefore $p_2^*$ is not a Nash equilibrium. The strategy profile $p_2^*$ is the unique Nash equilibrium of $G_c$, but is not an ESS\(^3\). The equilibrium $p_2^*$ is, however, an asymptotically stable rest point. The real part of the eigenvalues of the Jacobian at $p_2^*$ is $Re[A] = \frac{c}{25T}[RS - PT + c(P + R - S - T)] = \frac{c}{25T} \det[A_c]$. Since $S < 0$ then $sgn[Re[A]] = -sgn[det[A_c]]$.

  Since $R < c < \bar{c}$ then $\det[A_c] > 0$ and therefore $p_2^*$ is asymptotically stable. To see this, $(*) c < -P$ gives $c + P < 0$ and then $-T(c + P) > 0$. $(**)$ Since $c < \bar{c}$ then $RS - c(S - P - R) > 0$. Combining $(*)$ and $(**)$.

\(^2\)We discuss the properties of this game and its interior more thoroughly in Section 4

\(^3\)The reason why $p_2^*$ is not an ESS will be discussed in Section 4. We can also easily check that any strategy of the form $p^* = p_1^* + (\varepsilon, 0, -\varepsilon)$ earns the same payoff against itself as $p_3^*$ earns against it. This contradicts the second condition for evolutionary stability of $p_3^*$. 

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![Figure 9: The sign pattern of payoff matrix $A_c$](image)

$A_c = \begin{bmatrix} C & D & PC \\ C & 0 & - \\ D & + & 0 \\ PC & - & + 0 \end{bmatrix}$

![Figure 10: Phase diagram for the replicator dynamics](image)

<table>
<thead>
<tr>
<th>$(s_1, s_2) = (+, +)$</th>
<th>$(s_1, s_2) = (+, -)$</th>
<th>$(s_1, s_2) = (-, -)$</th>
<th>$(s_1, s_2) = (-, +)$</th>
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<tr>
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The reason why $p_2^*$ is not an ESS will be discussed in Section 4. We can also easily check that any strategy of the form $p^* = p_1^* + (\varepsilon, 0, -\varepsilon)$ earns the same payoff against itself as $p_3^*$ earns against it. This contradicts the second condition for evolutionary stability of $p_3^*$.
we get that $\det[A_c] > 0$. Along with $p^*_1$ and the three vertices of the simplex, $p^*_2$ is also a rest point of the replicator dynamics when cooperators are absent. For payoff entries $T = 3; R = 1; P = -3; S = -4$ and a cost $c = \frac{3}{2} \in (1, \bar{c} = 2)$, the phase diagram for the replicator dynamics is given in Figure 11.

Figure 11: Phase diagram for the replicator dynamics $|(s_1, s_2) = (+, +)| c \in (R, \bar{c}) | T = 3; R = 1; P = -3; S = -4$ and $c = \frac{3}{2}$

The last scenario where $(s_1, s_2) = (-, +)$ will be treated in the following section.

### 4. The rock-paper-scissors game $|(s_1, s_2) = (-, +)$

This occurs when $c < \min[R, -P]$. A necessary condition to have a positive cost is to have $P < 0$. Call $\underline{c} = \min[-P, R]$. If $(s_1, s_2) = (-, +)$, the sign pattern of $A_c$ is given by Figure 12.

$$
\begin{pmatrix}
C & D & PC \\
C & 0 & - & +
\end{pmatrix}
\begin{pmatrix}
D & + & 0 & - \\
PC & - & + & 0
\end{pmatrix}
$$

Figure 12: The sign pattern of payoff matrix $A_c | (s_1, s_2) = (-, +)$


**Theorem 2.** (Weissing, 1991, Theorem 5.6) Let $p^*$ be the unique fixed point of the rock-paper-scissors game $A_c$. The following holds true:

- $p^*$ is asymptotically stable in (3) if and only if $\det[A_c] > 0$. In this case, $p^*$ is even a global attractor for the continuous replicator dynamics.

- $p^*$ is unstable in (3) if and only if $\det[A_c] < 0$. In this case, $p^*$ is even a global repellor for the continuous replicator dynamics.
• $p^*$ is neutrally stable in (3) if and only if $\det[A_c] = 0$. In this case, $p^*$ is even a global centre for the continuous replicator dynamics.

**Theorem 3.** (Weissing, 1991, Corollary 4.5) Let $p^*$ an interior fixed point of a rock-papers-scissors game with payoff matrix in Figure 2(b). $p^*$ is an ESS then (1) $g_{C \rightarrow D} > 0$; (2) $g_{D \rightarrow C} > 0$; (3) $g_{C \rightarrow D} > 0$.

The unique equilibrium is the interior equilibrium $p^*_1$. The stability of $p^*_1$ depends on the sign of the determinant of $A_c$. We compute the determinant of $A_c$ and obtain:

$$\det[A_c] = c [RS - PT + c (P + R - S - T)]$$

Define $\alpha$ and $\beta$ such that $\alpha = P + R - S - T$ and $\beta = PT - RS$. It follows that $\det[A_c] = c (\alpha c - \beta)$. Note that if $\alpha = 0$, then $\text{sgn} [\det[A_c]] = -\text{sgn}[\beta]$. Assume now that $\alpha \neq 0$ and let $c^*$ be the cost such that $\det[A_c] = 0$. Since the derivative of the determinant with respect to $c$ has a fixed sign $\text{sgn}[\alpha]$, the determinant has different signs on the intervals $(0, c^*)$ and $(c^*, c)$. We find that $c^* < c$ if the following holds: (1) If $\alpha > 0$, then $S + T < R + P < 0$. If $\alpha < 0$ then $S + T > R + P > 0$. $c^* > 0$ if $\alpha \beta > 0$.

$$c^* = \frac{\beta}{\alpha} = \frac{PT - RS}{P + R - S - T}$$

We explicate below the conditions applying to the payoff structure and the cost $c$ for each stability scenario of the interior rest point (Theorem 2, Hofbauer and Sigmund, 1998, Theorem 7.7.2).

**The interior rest point $p^*$ is asymptotically stable if the determinant of the payoff matrix is positive. The trajectories of the replicator dynamics starting from any interior initial condition will converge to $p^*_1$ in an oscillatory fashion.**

The determinant of $A_c$ is positive if one of the following conditions is satisfied:

[A.1] $\alpha \geq 0$ and $\beta \leq 0$ with $(\alpha, \beta) \neq (0, 0)$

[A.2] $S + T < R + P < 0$ and $\beta > 0$

[A.3] $S + T > R + P > 0$ and $\beta < 0$

In condition A.1, $c^*$ is negative so that, for all $c \in (0, c^*)$, the population state converges to $p^*_1$.

**The interior rest point $p^*_1$ is a source i.e. all orbits in int($\Delta_3$) converge to a heteroclinic cycle, including the vertices of $\Delta_3$ and the edges connecting them, if the determinant of the payoff matrix is negative.**

The determinant of $A_c$ is negative if one of the following conditions is satisfied:

[B.1] $\alpha \leq 0$ and $\beta \geq 0$ with $(\alpha, \beta) \neq (0, 0)$

[B.2] $0 < P + R < S + T$ and $\beta < 0$

[B.3] $S + T < P + R < 0$ and $\beta > 0$

Condition B.1 ensures that the cutoff cost $c^*$ is negative. Since $\beta \geq 0$ then the determinant is negative.

**The interior rest point $p^*_1$ is a centre surrounded by periodic orbits if the determinant of the payoff matrix is zero.**
The determinant of $A_c$ is zero if one of the following conditions is satisfied:

- **[C.1]** $c \in (0, c^*)$
- **[C.2]** $c = c^*$ and $\alpha = 0$ and $\beta = 0$
- **[C.3]** $c > c^*$ and $\alpha > 0$ and $\beta > 0$

As an illustration, consider the following prisoners dilemma game with the payoff matrix $\begin{pmatrix} 3 & -4 \\ 7 & -2 \end{pmatrix}$. The parameters $\alpha$ and $\beta$ are such that $\alpha = P + R - S - T = -2$ and $\beta = PT - RS = -2$. We calculate the cut-off value $c^* = \frac{PT - RS}{P + R - S - T} = 1$. The value $c = \min[-P, R] = 2$. Depending on the cost $c$, we have three scenarios:

- $0 < c < 1$: The interior rest point $p_3^*$ is asymptotically stable (condition A.3.).
- $1 < c < 2$: The interior rest point $p_3^*$ is repelling and all orbits except at $p_3^*$ converge to the heteroclinic cycle formed by the vertices of $\Delta$ and the edges between them (condition B.2.)
- $c = c^* = 1$: $p_3^*$ is a centre surrounded by periodic orbits (condition C.2.).

![Phase diagrams](image)

The red color on Figure 13(a) denotes a higher speed. Prudent cooperators have a large advantage when they are present in only a small fraction of the population. They do better than both cooperators and defectors when meeting defectors and do not do much worse than them when meeting cooperators. When the population is composed of a majority of cooperators (C or PC), defection spreads slowly, given that it only has a relative advantage in meeting with cooperators. Recall that defection produces zero when prudent cooperators are met. This matches the intuition that defection spreads slowly in a population of cooperators until things get worse, and then cooperation spreads relatively quickly, due to awareness of the danger of defectors.

**Equilibrium payoff at $p_3^*$**: Recall that at the interior equilibrium, all strategies have the same payoff. We aim to verify the robustness of the equilibrium in two ways: (1) If the equilibrium payoff is negative; prudent cooperators might be better off refusing to play any game, ensuring thereby a payoff of zero. (2) Since $c < -P$ then defectors might do
we obtain:

$$\pi_C > 0 \Rightarrow R - c \left(1 - \frac{R}{S}\right) > 0 \Rightarrow c < \frac{RS}{S - R} \tag{8}$$

It follows that only for costs $c < \frac{RS}{S - R}$ that we expect prudent cooperators to engage in the game.

(2) We check if prudent defection $(PD)$ can invade $p_3^*$. Prudent defectors are defectors that pay a cost $c$ and abstain to play if they are matched with defectors. Note that a prudent defector earns $T - c$ against a cooperator and $-c$ against $D$ and $PC$. Call $\pi(x,y)$ the payoff that strategy $x$ earns against strategy $y$. We have then:

$$\pi(PD, p_3^*) = T \times \frac{RS - c(S - P - R)}{ST} - c \tag{9}$$

Since all strategies in the support of $p_3^*$ earn $\frac{RS+Rc}{S} - c$ against $p_3^*$; so $\pi(p_3^*, p_3^*) = \frac{RS+Rc}{S} - c$. The difference between $\pi(PD, p_3^*)$ and $\pi(p_3^*, p_3^*)$ is equal to $\pi(PD, p_3^*) - \pi(p_3^*, p_3^*) = \frac{(P-S)}{S} < 0$. Prudent defectors cannot invade the equilibrium strategy $p_3^*$.

If we restrict our attention to the case where $p_3^*$ is asymptotically stable and call $\hat{c} = \min\left(\frac{RS}{S-R}, c^e\right)$ and $\hat{c} = \min\left(\frac{RS}{S-R}, c\right)$. We obtain:

\[
\begin{align*}
[A.1] & \\
c \in (0, \hat{c}) & \quad [A.2] & \quad [A.3] \\
\alpha \geq 0 \text{ and } \beta \leq 0 \text{ with } (\alpha, \beta) \neq (0, 0) & \quad S + T < P + R < 0, \beta > 0 \text{ and } \hat{c} > c^* & \quad S + T > P + R > 0 \text{ and } \beta < 0
\end{align*}
\]

**Evolutionary stability of $p_3^*$**: Considering the payoff matrix 2(b) and using Theorem 3, we find that the interior equilibrium $p_3^*$ is never evolutionarily stable. This follows from the fact that $g_{C\rightarrow PC} = -g_{PC\rightarrow C}$, i.e. the gain from switching from being a prudent cooperator to being a cooperator is equal to the loss of switching from being a cooperator to being a prudent cooperator. Consequently, $p_3^*$ is not evolutionarily stable.

**Asymptotic stability of $p_3^*$**: Using the payoff matrix in Figure 2(b), the determinant of $A_c$ can be written as follows:

$$\det[A_c] = g_{C\rightarrow D} \times g_{D\rightarrow PC} \times g_{PC\rightarrow C} + g_{C\rightarrow PC} \times g_{PC\rightarrow D} \times g_{D\rightarrow C} \tag{10}$$

We have $g_{PC\rightarrow C} = -g_{C\rightarrow PC} = c$. It follows from the fact that the restriction of the game to the strategies $(C, PC)$ has an original payoff matrix of $\begin{pmatrix} R & R_C \\ R_P & R_{PC} \end{pmatrix}$, which has dimension 1 and determinant 0. Using $g_{PC\rightarrow C} = -g_{C\rightarrow PC} = c$, we obtain:

$$\det[A_c] = c \left[ g_{C\rightarrow D} \times g_{D\rightarrow PC} - g_{PC\rightarrow D} \times g_{D\rightarrow C} \right] \tag{11}$$

The stability of the interior equilibrium depends on the sign of $\det[A_c]$ i.e. on the difference between the two

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4Check also (Hofbauer and Sigmund, 1998, Exercise 7.7.1)
products of the gains of switching. Consider the two outcomes: the interior equilibrium and the boundary of \( \Delta \). Note that, at \( c = c^* \), the quantity \( Q(c) = g_{C \rightarrow D} \times g_{D \rightarrow PC} - g_{P \rightarrow D} \times g_{D \rightarrow C} = 0 \). This said, at \( c = c^* \), the replicator dynamics does not select any of the above outcomes. Assume that \( Q(c) \) is decreasing in \( c \). Then if \( c > c^* \) then the replicator dynamics selects the interior rest point.

Prisoners’ dilemma game with additive payoffs: We look at a special class of prisoners’ dilemma games, namely those with additive payoffs. Consider a prisoners’ dilemma game \( G \) with additive payoffs i.e. \( \exists (b, c') \in \mathbb{R}^2 \) with \( c' < b \) such that \( T = b, R = b - c', P = 0 \) and \( S = -c' \). For such games, \( \alpha = P + R - S - T = 0 \). Subtract from each entry in the payoff matrix a constant \( h \) with \( 0 < h < b - c' \), so as to have \( T > R > 0 > P > S \). We obtain \( T = b - h, R = b - c' - h, P = -h \) and \( S = -c' - h \). Clearly the game obtained still has additive payoffs and therefore \( \alpha = 0 \). The sign of the determinant depends solely on \( \beta \). In fact, \( \text{sgn} \{ \text{det}[A_c] \} = -\text{sgn}(\beta) \). We find that, if \( h > (b - c')/2 \) then \( \text{det}[A_c] > 0 \) and \( p^*_1 \) is asymptotically stable. If \( h < (b - c')/2 \) then \( \text{det}[A_c] < 0 \) and \( p^*_1 \) is a source. Finally, if \( h = (b - c')/2 \) then \( \text{det}[A_c] = 0 \) and \( p^*_1 \) is centre. A possible interpretation of the constant \( h \) is the following: Imagine business people pay a fee \( h \) to an intermediary to find business partners. Once they are matched, they enter a venture modelled by the prisoners’ dilemma game.

5. Conclusion

A French proverb says that \textit{Good nature without prudence is foolishness}. Our model says, however, that the burden of prudence doesn’t fall on the shoulders of every cooperator. It suffices that a share of cooperators becomes prudent so that defectors are deterred from taking over the population. Defectors do not go extinct because the cost of further reducing their fitness (beyond the equilibrium point) outweighs the resulting benefit for prudent cooperators. Figure 14 summarizes the results:

![Figure 14: Summary of the results of the analysis](image-url)

The cost of identifying the strategy of the opponent is crucial to the outcome of our model. Institutions provide individuals with information regarding their potential partners and clues on who to avoid. Recall that the equilibrium
fraction of defectors is positively correlated with cost. Consequently, the higher the efficiency of institutions in providing information at low cost, the higher the equilibrium fraction of cooperators (including the prudent cooperators) in society. Of course, it the cost is prohibitively high ($c > \tilde{c}$), then it is not worth for prudent cooperators to acquire information and act on it. Defection will take over the population and is evolutionarily stable. So no other strategy, in small fraction, will be able to change the state of the system.

If the cost is sufficiently low ($c < c^*$), the interior equilibrium where cooperators, prudent cooperators and defectors coexist, is asymptotically stable but not evolutionarily stable. The vulnerability to perturbations is actually a realistic feature. Indeed, in human societies, there occurs trends of defection e.g. pickpocketing, fake demands of money transfers, Ponzi schemes or rental scams. Formal or informal institutions usually react by providing information to citizens that allows them to avoid abuse. If information is available at a low cost, then it is expected that society regains its usual level of cooperation after a little while. Informed cooperators will block the way to defectors reducing their payoff bringing society back to equilibrium.

We used in a large part of the analysis the assumption that $P < 0$. There might be good reasons to use such an assumption. Even if the direct payoff related to the game itself is zero; we assume that there is a cost of playing the game (time, attention, cost of implementing the strategies), which would make the payoff of defectors negative. Prudent cooperators do not have to incur this cost because they avoid the game altogether. It seems therefore reasonable to assume that defectors get less than prudent cooperators when they meet defectors.

In our model, prudent informed cooperators just avoid to play defectors. This stands in clear contrast to punishment, considered as a major factor behind the evolution of cooperation. In fact, Dreber et al (2008) suggested that costly punishment was maladaptive and that winners do not punish. In many instances in real life, cooperators just avoid defectors. Examples, people do not respond to scam when looking for apartments, they do not give their financial information to a suspect individuals or organizations, they reject offers for risky joint work or venture and they refuse to help a beggar who seems to be lying. Information about trendy defective behavior allows individuals to avoid falling victims to defectors. However, it usually takes time until the information becomes available at a low cost, this creates the cyclical nature of the war between cooperators and defectors (Imhof et al (2005)). Defectors starts by spreading in society until they become so numerous that a prudent action is justified from a cost-benefit perspective (Prest and Turvey (1965)). In fact, Individuals do not usually rush to install an alarm as soon as they hear about a theft in the neighborhood or do they count every penny they receive from a cashier if it happens that he miscalculates once. It is only when such instances become frequent that corrective measures are justified. Once the prudent cooperators take these measures, the payoff of the defectors is reduced and the interior equilibrium is reached again. The role of institutions is then to produce information at a sufficiently low cost and make it available to cooperators, so as to stop the advances of defectors and reduce their equilibrium fraction.
References


14